Dr. G.S Prasad M.Sc., M.Phil., Ph.D.
Professor, Department of Mathematics
C.M.R Institute of technology
No.132, AECS Layout, ITPL Road
Bangalore – 560037
Email:-gsprasadphd@gmail.com

Gradient of a scalar function, unit normal, directional derivative, divergence of a vector function, Curl of a vector function, solenoidal and irrotational fields, simple and direct problems, application of Laplace transform to differential equation and simultaneous differential equations.

Gradient of a Scalar Function

Let $\phi = \phi(x,y,z)$ be a given scalar field, then the vector whose X, Y, Z components are $\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}$ respectively is called the gradient of ϕ at the point (x,y,z). It is denoted by grad ϕ or $\nabla\phi$.

Grad $\phi = \nabla\phi = \frac{\partial \phi}{\partial x}\hat{\imath} + \frac{\partial \phi}{\partial y}\hat{\jmath} + \frac{\partial \phi}{\partial z}\hat{k}$

Problems:-

1. If $\phi(x,y,z) = x^2 - y^2 + 2yz + 2z^2$ find $\nabla\phi$ at the point (1,-2,1).

   $\nabla\phi = \frac{\partial \phi}{\partial x}\hat{\imath} + \frac{\partial \phi}{\partial y}\hat{\jmath} + \frac{\partial \phi}{\partial z}\hat{k}$

   $\nabla\phi = 2x\hat{\imath} + (-2y + 2z)\hat{\jmath} + (2y + 4z)\hat{k}$

   $\nabla\phi(1,-2,1) = 2\hat{\imath} + 6\hat{\jmath} + 0\hat{k}$

2. If ϕ(x,y,z) = $x^2y^3z^4$ find grad ϕ.

$\nabla\phi = \frac{\partial\phi}{\partial x}\,\hat{\imath} + \frac{\partial\phi}{\partial y}\,\hat{\jmath} + \frac{\partial\phi}{\partial z}\,\hat{k}$

$= 2xy^3z^4\hat{\imath} + 3x^2y^2z^4\hat{\jmath} + 4x^2y^3z^3\hat{k}$

3. If ϕ = $xy + yz + zx$ and $\vec{A} = x^2y\hat{\imath} + y^2z\hat{\jmath} + z^2x\hat{k}$ show that

$\vec{A}$.gradϕ $= 25$ at the point (3,-1,2)

$\nabla\phi = \frac{\partial\phi}{\partial x}\,\hat{\imath} + \frac{\partial\phi}{\partial y}\,\hat{\jmath} + \frac{\partial\phi}{\partial z}\,\hat{k} = (y+z)\hat{\imath} + (x+z)\hat{\jmath} + (y+x)\hat{k}$

$\nabla\phi(3,-1,2) = \hat{\imath} + 5\hat{\jmath} + 2\hat{k}$ -------- 1

$\vec{A} = -9\hat{\imath} + 2\hat{\jmath} + 12\hat{k}$ --------------- 2

$\vec{A}$. gradϕ $= -9 + 10 + 24 = 25$

4. If ϕ = $x^3 + y^3 + z^3 - 3xyz$ find ∇ϕ and |∇ϕ| at the point p(1,-1,2).

$\nabla\phi = \frac{\partial\phi}{\partial x}\,\hat{\imath} + \frac{\partial\phi}{\partial y}\,\hat{\jmath} + \frac{\partial\phi}{\partial z}\,\hat{k}$

$\nabla\phi = (3x^2 - 3yz)\hat{\imath} + (3y^2 - 3xz)\hat{\jmath} + (3z^2 - 3xy)\hat{k}$

$\nabla\phi(1,-1,2) = 9\hat{\imath} - 3\hat{\jmath} + 15\hat{k}$

$|\nabla\phi|\,(1,-1,2) = \sqrt{(9)^2 + (-3)^2 + (15)^2} = 3\sqrt{25}$

5. If $\phi = x^2y + y^2z + z^2x$ find $\nabla\phi$ at (1,2,3)

$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{\imath} + \frac{\partial\phi}{\partial y}\hat{\jmath} + \frac{\partial\phi}{\partial z}\hat{k}$

$\nabla\phi = (z^2 + 2xy)\hat{\imath} + (x^2 + 2yz)\hat{\jmath} + (y^2 + 2zx)\hat{k}$

$\nabla\phi(1,2,3) = 13\hat{\imath} + 13\hat{\jmath} + 10\hat{k}$

6. If $\phi = x^n + y^n + z^n$ where n is a non-zero real constant, prove that $\nabla\phi . r = n\phi$

$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{\imath} + \frac{\partial\phi}{\partial y}\hat{\jmath} + \frac{\partial\phi}{\partial z}\hat{k}$

$= nx^{n-1}\hat{\imath} + ny^{n-1}\hat{\jmath} + nz^{n-1}\hat{k}$

$\nabla\phi . r = n\left(x^{n-1}\hat{\imath} + y^{n-1}\hat{\jmath} + z^{n-1}\hat{k}\right).\left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right)$

$= n(x^n + y^n + z^n) = n\phi$

7. If a is a constant vector show that $\nabla(a.\vec{r}) = a$

Let $a = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$, $r = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$a.\vec{r} = a = a_1x + a_2y + a_3z$

$\nabla(a.\vec{r}) = \frac{\partial(a.\vec{r})}{\partial x}\hat{\imath} + \frac{\partial(a.\vec{r})}{\partial y}\hat{\jmath} + \frac{\partial(a.\vec{r})}{\partial z}\hat{k}$

$= a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k} = a$

Unit Normal:- $\nabla\phi$ is the normal vector to the surface $\phi(x,y,z)$ then a unit normal vector is denoted by $\hat{n}$ is defined as

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\vec{a}}{|\vec{a}|}$$

Where $\vec{a} = \nabla\phi$ = normal vector

Directional Derivative:- if $\vec{a}$ is any vector and ϕ is any scalar point function then $\nabla\phi.\frac{\vec{a}}{|\vec{a}|}$ represents the component of $\nabla\phi$ in the direction of $\vec{a}$ which is known as the directional derivative of ϕ in the direction of $\vec{a}$.

Directional derivative $= \nabla\phi.\frac{\vec{a}}{|\vec{a}|} = \nabla\phi.\hat{a}$

Problems:-

1. Find the unit normal to the surface $yz + zx + xy = c$ at the point $p(-1,2,3)$.

The equation of the given surface is $\phi(x,y,z) = c$

$$\phi = yz + zx + xy$$

$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{\imath} + \frac{\partial\phi}{\partial y}\hat{\jmath} + \frac{\partial\phi}{\partial z}\hat{k}$

$$\nabla\phi = (y+z)\hat{\imath} + (x+z)\hat{\jmath} + (y+x)\hat{k}$$

$\nabla\phi(-1,2,3) = 5\hat{\imath} + 2\hat{\jmath} + \hat{k},$

$|\nabla\phi| = \sqrt{25+4+1} = \sqrt{30}$

$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{5\hat{\imath}+2\hat{\jmath}+\hat{k}}{\sqrt{30}}$

2.Find the unit normal to the surface $x^3y^3z^2 = 4$ at the point $p(-1,-1,2)$

The equation of the given surface is $\phi(x,y,z) = -4 + x^3y^3z^2$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{\imath} + \frac{\partial\phi}{\partial y}\hat{\jmath} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$= 3x^2y^3z^2\hat{\imath} + 3x^3y^2z^2\hat{\jmath} + 2x^3y^3z\hat{k}$$

$$\nabla\phi(-1,-1,2) = -12\hat{\imath} - 12\hat{\jmath} + 4\hat{k}$$

$$|\nabla\phi| = \sqrt{(-12)^2 + (-12)^2 + 4^2} = \sqrt{304}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{-12\hat{\imath}-12\hat{\jmath}+4\hat{k}}{\sqrt{304}}$$

$$= \frac{-4}{\sqrt{304}}(3\hat{\imath} + 3\hat{\jmath} - \hat{k})$$

3.Find the angle between the direction of the normals to the surface $x^2yz = 1$

at the point $p(-1,1,1)$ and $q(1,-1,-1)$

The given surface is $\phi(x,y,z) = x^2yz = 1$

At any point (x,y,z) of this surface the normal is along the vector.

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{\imath} + \frac{\partial\phi}{\partial y}\hat{\jmath} + \frac{\partial\phi}{\partial z}\hat{k} = 2xyz\hat{\imath} + x^2z\hat{\jmath} + x^2y\hat{k}$$

Therefore, at the point $p(-1,1,1)$ the normal is along the vector

a = $|\nabla\phi|p = -2\hat{\imath} + \hat{\jmath} + \hat{k}$

at the point q$(1, -1, -1)$ the normal is along the vector

b = $|\nabla\phi|q = 2\hat{\imath} - \hat{\jmath} - \hat{k}$

If θ is the angle between the direction of these normals, we have

$$\cos\theta = \frac{a \cdot b}{|a||b|} = \frac{-6}{\sqrt{6}\sqrt{6}} = -1$$

This gives $\theta = \pi$ as the required angle. Thus at the given points the normals to the given surfaces are in opposite direction.

4.Find the angle between the surfaces

$x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $p(2, -1, 2)$

The angle between the surfaces at a common point p is defined to be equal to the angle between the normals to the surface at the point p.

The given surfaces are S_1 whose equation is $\phi(x, y, z) = x^2 + y^2 + z^2 = 9$

And S_2 whose equation is $\psi(x, y, z) = x^2 + y^2 - z = 3$

$\nabla\phi = 2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{k} = 4\hat{\imath} - 2\hat{\jmath} + 2\hat{k}$

$\nabla\psi = 2x\hat{\imath} + 2y\hat{\jmath} - \hat{k} = 4\hat{\imath} - 2\hat{\jmath} - \hat{k}$

$\nabla\phi$ is along the normal to the surface S_1 and $\nabla\psi$ is along the normal to the surface S_2 . Therefore, θ is the angle between the surfaces S_1 and S_2 at the point p, then θ is the angle between $\nabla\phi$ and $\nabla\psi$ at p.

$\nabla\phi . \nabla\psi = |\nabla\phi||\nabla\psi|\cos\theta$

$$\cos\theta = \frac{\nabla\phi . \nabla\psi}{|\nabla\phi||\nabla\psi|} = \frac{2(2\hat{\imath} - \hat{\jmath} + \hat{k}).(4\hat{\imath} - 2\hat{\jmath} - \hat{k})}{\sqrt{16+4+4}\ \sqrt{16+4+1}}$$

$= \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$ $\qquad \theta = \cos^{-1}\frac{8}{3\sqrt{21}}$

5.Find the angle between the tangents to the curve

$\vec{r} = \left(t - \frac{t^3}{3}\right)\hat{\imath} + t^2\hat{\jmath} + \left(t + \frac{t^3}{3}\right)\hat{k}$ at $t = \pm 3$

$$\left(\frac{d\vec{r}}{dt}\right) = \left(1 - \frac{3t^2}{3}\right)\hat{\imath} + 2t\hat{\jmath} + \left(1 + \frac{3t^2}{3}\right)\hat{k}$$

When $t = 3$, $t_1 = \left(\frac{d\vec{r}}{dt}\right) = (1-9)\hat{\imath} + 6\hat{\jmath} + (1+9)\hat{k} = -8\hat{\imath} + 6\hat{\jmath} + 10\hat{k}$

When $t = -3$, $t_2 = \left(\frac{d\vec{r}}{dt}\right) = (1-9)\hat{\imath} - 6\hat{\jmath} + (1+9)\hat{k} =$

$$-8\hat{\imath} - 6\hat{\jmath} + 10\hat{k}$$

$$\cos\theta = \frac{t_1 . t_2}{|t_1||t_2|} = \frac{(-8\hat{\imath} + 6\hat{\jmath} + 10\hat{k}).(-8\hat{\imath} - 6\hat{\jmath} + 10\hat{k})}{\sqrt{64+36+100}\sqrt{64+36+100}}$$

$$= \frac{64 - 36 + 100}{\sqrt{200}\,\sqrt{200}} = \frac{128}{200} = \frac{16}{25}$$

6. Find the directional derivation of $\phi = x^2yz + 4xz^2$ at $(1, -2, -1)$ along $2\hat{\imath} - \hat{\jmath} - 2\hat{k}$.

$$\nabla\phi = (2xyz + 4z^2)\hat{\imath} + x^2z\hat{\jmath} + (x^2y + 8xz)\hat{k}$$

$\nabla\phi(1, -2, -1) = 8\hat{\imath} - \hat{\jmath} - 10\hat{k}$ ----------------1

$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{2\hat{\imath} - \hat{\jmath} - 2\hat{k}}{\sqrt{4+1+4}} = \frac{2\hat{\imath} - \hat{\jmath} - 2\hat{k}}{3}$ ----------------2

$$\nabla\phi . \hat{a} = \frac{1}{3}[16 + 1 + 20] = \frac{37}{3}$$

7.Find the directional derivative of $\phi(x, y, z) = 2x^2y^3z^4$ at $(1, -1, 1)$ in the direction of $\hat{\imath} + 2\hat{\jmath} - 2\hat{k}$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{\imath} + \frac{\partial\phi}{\partial y}\hat{\jmath} + \frac{\partial\phi}{\partial z}\hat{k} =$$

$$4xy^3z^4\hat{\imath} + 6x^2y^2z^4\hat{\jmath} + 8x^2y^3z^3\hat{k}$$

$$\nabla\phi(1, -1, 1) = -4\hat{\imath} + 6\hat{\jmath} - 8\hat{k}$$

$$\hat{a} = \frac{\hat{\imath} + 2\hat{\jmath} - 2\hat{k}}{\sqrt{9}} = \frac{\hat{\imath} + 2\hat{\jmath} - 2\hat{k}}{3}$$

$$\nabla\phi.\hat{a} = \frac{1}{3}(-4\hat{\imath} + 6\hat{\jmath} - 8\hat{k}).(\hat{\imath} + 2\hat{\jmath} - 2\hat{k})$$

$$= \frac{1}{3}[-4 + 12 + 16] = \frac{24}{3} = 8$$

8.Find the directional derivative of $\phi(x, y, z) = x^2yz + xz^2$ at (-1,2,1) in the direction of $2\hat{\imath} - \hat{\jmath} - 2\hat{k}$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{\imath} + \frac{\partial\phi}{\partial y}\hat{\jmath} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$= (2xyz + z^2)\hat{\imath} + x^2z\hat{\jmath} + (x^2y + 2zx)\hat{k}$$

$\nabla\phi(-1, 2, 1) = -3\hat{\imath} + \hat{\jmath} + 0\hat{k}$ $\qquad \hat{a} = \frac{2\hat{\imath} - \hat{\jmath} - 2\hat{k}}{3}$

$$\nabla\phi.\hat{a} = \frac{1}{3}[-6 - 1] = \frac{-7}{3}$$

9. If $\phi = \frac{xz}{x^2+y^2}$ find the directional derivative at (1,-1,1) in the direction $\vec{A} = \hat{\imath} - 2\hat{\jmath} + \hat{k}$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{\imath} + \frac{\partial\phi}{\partial y}\hat{\jmath} + \frac{\partial\phi}{\partial z}\hat{k}$$

$\frac{\partial \phi}{\partial x} = \frac{y^2z - zx^2}{(x^2+y^2)^2}$ $\frac{\partial \phi}{\partial y} = \frac{-2xzy}{(x^2+y^2)^2}$ $\frac{\partial \phi}{\partial z} = \frac{x}{(x^2+y^2)}$

$\nabla\phi = \frac{y^2z - zx^2}{(x^2+y^2)^2}\hat{\imath} + \frac{-2xzy}{(x^2+y^2)^2}\hat{\jmath} + \frac{x}{(x^2+y^2)}\hat{k}$

$\nabla\phi(1,-1,1) = 0\hat{\imath} + \frac{2}{4}\hat{\jmath} + \frac{1}{2}\hat{k} = \frac{1}{2}\hat{\jmath} + \frac{1}{2}\hat{k}$

$$\hat{a} = \frac{\hat{\imath} - 2\hat{\jmath} + \hat{k}}{\sqrt{6}}$$

$\nabla\phi.\hat{a} = \frac{1}{\sqrt{6}}\left[-1 + \frac{1}{2}\right] = \frac{-1}{2\sqrt{6}}$

10. If the directional derivative of $\phi = axy^2 + byz + cz^2x^3$ at the point $p(-1,1,2)$ has a maximum magnitude of 32 units in the direction parallel to the y-axis, find a,b,c.

$\nabla\phi = (\mathrm{ay}^2 + 3\mathrm{cz}^2\mathrm{x}^2)\hat{\imath} + (2axy + bz)\hat{\jmath} +$

$$(by + 2czx^3)\hat{k}$$

$\nabla\phi(-1,1,2) = (a + 12c)\hat{\imath} + (-2a + 2b)\hat{\jmath} +$

$(b - 4c)\hat{k}$ -------1

It is given that the directional derivative of ϕ at p has a maximum magnitude of 32 in the direction parallel to the y-axis $\nabla\phi = 32\hat{\jmath}$ ------------2

Compare 1 & 2

$$-2a + 2b = 32 \qquad a + 12c = 0 \qquad b - 4c = 16$$

$$(-a + b = 16) + (a + 12c = 0) = (b + 12c = 16)$$

$$(b + 12c = 16) - (b - 4c = 0) = (16c = 16)$$

$$c = 1,\ b = 4,\ a = -12$$

11.Find the maximum directional derivative of $\log(x^2 + y^2 + z^2)$ at (1,1,1)

The maximum directional derivative of ϕ is $|\nabla\phi|$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\,\hat{\imath} + \frac{\partial\phi}{\partial y}\,\hat{\jmath} + \frac{\partial\phi}{\partial z}\,\hat{k}$$

$$= \left(\frac{2x}{x^2 + y^2 + z^2}\right)\hat{\imath} + \left(\frac{2y}{x^2 + y^2 + z^2}\right)\hat{\jmath} + \left(\frac{2z}{x^2 + y^2 + z^2}\right)\hat{k}$$

$$\nabla\phi(1,1,1) = \frac{2}{3}\hat{\imath} + \frac{2}{3}\hat{\jmath} + \frac{2}{3}\hat{k} \qquad |\nabla\phi| = \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{4}{9}} = \frac{6}{3} = 2$$

12.Find the angle between the surfaces $x^2yz + 3xz = 5$ and $x^2yz^3 = 2$ at (1,-2,1)

$$S_1 = \nabla\phi = \frac{\partial\phi}{\partial x}\,\hat{\imath} + \frac{\partial\phi}{\partial y}\,\hat{\jmath} + \frac{\partial\phi}{\partial z}\,\hat{k}$$

$$= (2xyz + 3z^2)\hat{\imath} + x^2z\hat{\jmath} + (x^2y + 6zx)\hat{k}$$

$$\nabla\phi = (4 + 3)\hat{\imath} - \hat{\jmath} + (-2 - 6)\hat{k} = 7\hat{\imath} - \hat{\jmath} - 8\hat{k}$$

$$S_2 = \nabla\psi = 2xyz^3\hat{\imath} + x^2z^3\hat{\jmath} + 3x^2yz^2\hat{k}$$

$$\nabla\psi = 4\hat{\imath} - \hat{\jmath} - 6\hat{k}$$

$$\cos\theta = \frac{\nabla\phi\,.\,\nabla\psi}{|\nabla\phi||\nabla\psi|} = \frac{(7\hat{\imath} - \hat{\jmath} - 8\hat{k}).(4\hat{\imath} - \hat{\jmath} - \hat{k})}{\sqrt{49 + 1 + 64}\ \sqrt{16 + 1 + 36}}$$

$$= \frac{28 + 1 + 48}{\sqrt{114}\,\sqrt{53}}$$

Divergence of a vector function :-

If $\vec{A} = A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k}$ is a vector function defined and differentiable at each point (x,y,z) then divergence of $\vec{A}$ is denoted by div $\vec{A}$ or $\nabla.\vec{A}$ and is defined by div $\vec{A} = \nabla.\vec{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$

Hence, divergence factor of a vector function is a scalar function.

Irrotational vector or conservative force field or potential field :-

A vector field $\vec{A}$ is said to be a irrotational vector or a conservative force field or potential field or curl force vector if $\nabla X\vec{A} = 0$

Scalar potential:- a vector field $\vec{A}$ which can be derived from the scalar field ϕ such that F= $\nabla\phi$ is called conservative force field and ϕ is called Scalar potential.

1.Show that $\vec{F} = \frac{x\hat{\imath}+y\hat{\jmath}}{x^2+y^2}$ is both solenoidal and irrotational.

div$\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \frac{-x^2+y^2}{(x^2+y^2)} + \frac{x^2-y^2}{(x^2+y^2)} = 0$

$$\text{Curl } \vec{F} = \nabla X\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} & 0 \end{vmatrix}$$

$$= \hat{\imath}\left[0 - \frac{\partial}{\partial z}\left(\frac{y}{x^2+y^2}\right)\right] - \hat{\jmath}\left[0 - \frac{\partial}{\partial z}\left(\frac{x}{x^2+y^2}\right)\right] + \hat{k}\left[\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)\right]$$

$$0\hat{\imath} + \hat{\jmath} + \hat{k}\left[\frac{-2yx}{(x^2+y^2)^2} - \frac{-2yx}{(x^2+y^2)^2}\right]$$

= 0 is irrotational.

2.Find the constants a,b,c such that the vector $\vec{F} = (siny + az)\hat{\imath} + (bxcosy + z)\vec{j} + (x + cy)\hat{k}$ is irrotational

$$\nabla X\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ siny + az & bxcosy + z & x + cy \end{vmatrix}$$

$$= \hat{\imath}\left[\frac{\partial}{\partial y}(x + cy) - \frac{\partial}{\partial z}(bxcosy + z)\right] - \hat{\jmath}\left[\frac{\partial}{\partial x}(x + cy) - \frac{\partial}{\partial z}(siny + az)\right] + \hat{k}\left[\frac{\partial}{\partial x}(bxcosy + z) - \frac{\partial}{\partial y}(siny + az)\right]$$

$=\hat{\imath}[c - 1] - \hat{\jmath}[1 - a] + \hat{k}[bcosy - cosy]$ If a=b=c=1,

$=\hat{\imath}[1 - 1] - \hat{\jmath}[1 - 1] + \hat{k}[cosy - cosy] = 0$

3.Find the divergence and curl of the vector

$$\vec{F} = (3x^2y - z)\hat{\imath} + (xz^2 + y^4)\hat{\jmath} - 2x^2z^2\hat{k}$$

Div $\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 6xy + 4y^3 - 4x^2z$

$$\nabla X\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2y - z & xz^2 + y^4 & -2x^2z^2 \end{vmatrix}$$

$=\hat{\imath}[0 - 2zx] - \hat{\jmath}[-4xz^2 + 1] + \hat{k}[z^2 + 3x^2]$ $=-2zx\hat{\imath} - [1 - 4xz^2]\hat{\jmath} + [z^2 - 3x^2]\hat{k}$

4. If $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ show that (i) $\nabla\vec{r} = 3$ (ii) $\nabla X\vec{r} = 0$

Div $\vec{r} = \frac{\partial r_1}{\partial x} + \frac{\partial r_2}{\partial y} + \frac{\partial r_3}{\partial z} = 1 + 1 + 1 = 3$

Curl $\vec{F} = \nabla X\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & 1 & 1 \end{vmatrix}$

$=\hat{\imath}[0-0] - \hat{\jmath}[0-0] + \hat{k}[0-0] = 0$

5.Find the constants a,b,c such that the vector field $\vec{F} = (x + y + az)\hat{\imath} + (bx + 2y - z)\hat{\jmath} + (x + cy + 2z)\hat{k}$ is irrotational

$\nabla X\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + y + az & bx + 2y - z & x + cy + 2z \end{vmatrix}$

$=\hat{\imath}[c-1] - \hat{\jmath}[1-a] + \hat{k}[b-1]$ If c=-1, a=1, b=1

$=\hat{\imath}[-1-1] - \hat{\jmath}[1-1] + \hat{k}[1-1] = 0$

6. $\vec{F} = e^{xyz}(\hat{\imath} + \hat{\jmath} + \hat{k})$ find div$\vec{F}$ and curl$\vec{F}$

Div $\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = e^{xyz}yz + e^{xyz}xz + e^{xyz}xy$

Curl $\vec{F} = \nabla X\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{xyz} & e^{xyz} & e^{xyz} \end{vmatrix}$

$=\hat{\imath}[e^{xyz}xz - e^{xyz}xy] - \hat{\jmath}[e^{xyz}yz - e^{xyz}xy]$

$+\hat{k}[e^{xyz}yz - e^{xyz}xz]$

7.If $\vec{V} = 3xy\hat{\imath} + 3x^2y\hat{\jmath} - 3ayz\hat{k}$ is solenoidal at (1,1,1) find a

Div$\vec{V} = \nabla.\vec{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$

$$= 3y + 3x^2 - 3ay$$

If a=2, Div$\vec{V} = \nabla.\vec{V} = 3 + 3 - 3(2) = 0$

is solenoidal.

8.Find div$\vec{A}$ and curl$\vec{A}$ where $\vec{A} = grad(x^3 + y^3 + z^3 - 3xyz)$

$\vec{A} = (3x^2 - 3yz)\hat{\imath} + (3y^2 - 3xz)\hat{\jmath} + (3z^2 - 3xy)\hat{k}$

div$\vec{A} = 6x + 6y + 6z$

Curl $\vec{A}$ = $\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 - 3yz & 3y^2 - 3xz & 3z^2 - 3xy \end{vmatrix}$

=$\hat{\imath}[-3x + 3x] - \hat{\jmath}[-3y + 3y] + \hat{k}[-3z + 3z] = 0$

9.Find the constant ‘a’ so that $\vec{A} = y(ax^2 + z)\hat{\imath} + x(y^2 - z^2)\hat{\jmath} + 2xy(z - xy)\hat{k}$ is solenoidal

div$\vec{A} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

$$= 2axy + 2xy + 2xy = 2axy + 4xy$$

$= -4xy + 4xy = 0$ is solenoidal

10. If $\vec{F} = (y+z)\hat{\imath} + (z+x)\hat{\jmath} + (x+y)\hat{k}$ is irrational. Also find a scalar function such that $\vec{F} = \nabla\phi$

$$\nabla X\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix}$$

$\hat{\imath}[1-1] - \hat{\jmath}[1-1] + \hat{k}[1-1] = 0$ is irrotational

$\vec{F} = \nabla\phi$

$$(y+z)\hat{\imath} + (z+x)\hat{\jmath} + (x+y)\hat{k} = \frac{\partial\phi}{\partial x}\hat{\imath} + \frac{\partial\phi}{\partial y}\hat{\jmath} + \frac{\partial\phi}{\partial z}\hat{k}$$

$\frac{\partial\phi}{\partial x} = y+z \quad \frac{\partial\phi}{\partial y} = z+x \quad \frac{\partial\phi}{\partial z} = x+y$

$$\phi = \int (y+z)dx + f_1(y,z) = (y+z)x + f_1(y,z)$$

$$\phi = \int (z+x)dy + f_2(x,z) = (z+x)y + f_2(x,z)$$

$$\phi = \int (x+y)dz + f_3(x,y) = (x+y)z + f_3(x,y)$$

$$\phi = (y+z)x + (z+x)y + (x+y)z$$

11. $\vec{F} = x^2\hat{\imath} + y^2\hat{\jmath} + z^2\hat{k}$ then find (i) $\nabla.\vec{F}$ (ii) $\nabla X\vec{F}$ at the point (1,1,1)

$\nabla.\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 2x + 2y + 2z = 6$

$$\nabla X\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$= \hat{\imath}[0-0] - \hat{\jmath}[0-0] + \hat{k}[0-0] = 0$

12. Find divergence and curl of the vector

$$\vec{F} = (xyz + y^2z)\hat{\imath} + (3x^2 + y^2z)\hat{\jmath} + (xz^2 - y^2z)\hat{k}$$

div$\vec{F} = yz + 2yz + (2x^2 - y^2) = 3yz + 2xz - yz$

$$\nabla X\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz + y^2z & 3x^2 + y^2z & xz^2 - y^2z \end{vmatrix}$$

$$= \hat{\imath}[-2yz - y^2] - \hat{\jmath}[z^2 - xy - y^2] + \hat{k}[6x - xz - 2yz]$$

13.If $\vec{F} = (x + y + 1)\hat{\imath} + \hat{\jmath} - (x + y)\hat{k}$ show that $\vec{F}.curl\vec{F} = 0$

$$\nabla X\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x + y + 1) & 1 & -(x + y) \end{vmatrix}$$

$$= \hat{\imath}[-1 - 0] - \hat{\jmath}[-1 - 0] + \hat{k}[0 - 1] = -\hat{\imath} + \hat{\jmath} - \hat{k}$$

$$\vec{F}.curl\vec{F} = [(x + y + 1)\hat{\imath} + \hat{\jmath} - (x + y)\hat{k}].(-\hat{\imath} + \hat{\jmath} - \hat{k})$$

$$\vec{F}.curl\vec{F} = -(x + y + 1) + 1 + (x + y) = 0$$

14.Find the constants a,b and c so that the

vector $\vec{F} = (x + 2y + az)\hat{\imath} + (bx - 3y - z)\hat{\jmath}$

$+(4x + cy + 2z)\hat{k}$ is irrotational

$$\nabla X\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix}$$

$\nabla X\vec{F} = \hat{\imath}[c + 1] - \hat{\jmath}[4 - a] + \hat{k}[b - 2]$

If c=-1, a=4, b=2 $\quad\quad$ $\nabla X\vec{F} = 0$

15. Find the value of a,b,c for which the vector

$\vec{V} = (x + y + az)\hat{\imath} + (bx + 3y - z)\hat{\jmath} + (3x + cy + 2z)\hat{k}$ is irrotational

$$\nabla X \vec{V} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + y + az & bx + 3y - z & 3x + cy + 2z \end{vmatrix}$$

$$\nabla X \vec{V} = \hat{\imath}[c + 1] - \hat{\jmath}[3 - a] + \hat{k}[b - 1]$$

If c=-1, a=3, b=1 $\nabla X \vec{V}$ =0

16. Find $curl(curl\vec{A})$ given $\vec{A} = xy\hat{\imath} + y^2\hat{\jmath} + z^2y\hat{k}$

$$curl\vec{A} = \nabla X\vec{A} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & y^2z & z^2x \end{vmatrix}$$

$$= \hat{\imath}[z^2 - y^2] - \hat{\jmath}[0 - 0] + \hat{k}[0 - x] = [z^2 - y^2]\hat{\imath} + 0\hat{\jmath} + x\hat{k}$$

$$curl(curl\vec{A}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 - y^2 & 0 & x \end{vmatrix}$$

$$= \hat{\imath}[0 - 0] - \hat{\jmath}[-1 - 2z] + \hat{k}[0 - 2y] = 0\hat{\imath} + [1 + 2z]\hat{\jmath} + 2y\hat{k}$$

17. $\vec{F} = (3x^2y - z)\hat{\imath} + (xz^3 + y^4)\hat{\jmath} - 2x^3z^2\hat{k}$ find $grad\ (div\vec{F})$ at (2,-1,0)

$$div\vec{F} = 6xy + 4y^3 - 4x^3z$$

$$grad(div\vec{F}) = (-6 - 0)\hat{\imath} + (12 + 12)\hat{\jmath} - 32\hat{k}$$

$$= -6\hat{\imath} + 24\hat{\jmath} - 32\hat{k}$$

18.For what value of ‘a’ does the vector

$\vec{F} = (ax^2y + yz)\hat{\imath} + (xy^2 - xz^2)\hat{\jmath} + (2xyz - 2x^2y^2)\hat{k}$

has zero divergences. Also find $\nabla X\vec{F}$

$div\,\vec{F} = 2axy + 2yx + 2xy$

if a=-2

$= -4xy + 4xy = 0$

$$\nabla X\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ax^2y + y & xy^2 - xz^2 & 2xyz - 2x^2y^2 \end{vmatrix}$$

$$= \hat{\imath}[2xz - 4x^2y\,] - \hat{\jmath}[2yz - 4xy^2] + \hat{k}[(y^2 - z^2) - (ax^2 + z)]$$

19.If $\vec{F} = 3x^2\hat{\imath} + 5xy^2\hat{\jmath} + xyz^3\hat{k}$ at (1,2,3), find div$\vec{F}$

$$\nabla.\vec{F} = 6x + 10xy + 3xyz^2$$

$$\nabla.\vec{F}(1,2,3) = 6 + 20 + 54 = 80$$

20.Find $curl[xyz\hat{\imath} + 3x^2y\hat{\jmath} + (xz^2 - y^2z)\hat{k}]$

$$curl\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix}$$

=$\hat{\imath}[-2yz - 0] - \hat{\jmath}[z^2 - xy] + \hat{k}[6xy - xz]$

21.Find the divergence and the curl of the vector $\vec{F} = (xyz + y^2z)\hat{\imath} + (3x^2y + y^2z)\hat{\jmath} + (xz^2 - y^2z)\hat{k}$

$$div\vec{F} = yz + (3x^2 + 2yz) + 2xz - y^2$$

$$curl\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz + y^2z & 3x^2y + y^2z & xz^2 - y^2z \end{vmatrix}$$

$$= \hat{\imath}[-2yz - y^2] - \hat{\jmath}[z^2 - xy - y^2] + \hat{k}[6xy - xz - 2yz]$$

22.Prove that $div(curl\vec{A}) = 0$ if $\vec{A} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

$$curl\vec{A} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right)\hat{\imath} - \left(\frac{\partial a_3}{\partial x} - \frac{\partial a_1}{\partial z}\right)\hat{\jmath} + \left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y}\right)\hat{k}$$

23. Prove that $curl(grad\phi) = 0$

$$grad\phi = \frac{\partial\phi}{\partial x}\,\hat{\imath} + \frac{\partial\phi}{\partial y}\,\hat{\jmath} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$curl(grad\phi) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \hat{\imath}\left[\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial y\partial z}\right] - \hat{\jmath}\left[\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial x\partial z}\right] + \hat{k}\left[\frac{\partial^2\phi}{\partial y\partial x} - \frac{\partial^2\phi}{\partial y\partial x}\right] = 0$$

24.Prove that $\nabla(\vec{A}\,X\,\vec{B}) = \vec{B}.curl\vec{A} - \vec{A}.curl\vec{B}$

$\nabla(\vec{A}\,X\,\vec{B}) = \Sigma\hat{\imath}.\frac{\partial}{\partial x}(A\,X\,B)$

$$= \Sigma\hat{\imath}.\left(\frac{\partial A}{\partial x}\,X\,B\right) + \Sigma\hat{\imath}.(A\,X\frac{\partial B}{\partial x})$$

Interchange dot to cross

$$= \Sigma\left(\hat{\imath}\,X\frac{\partial A}{\partial x}\right).B - \Sigma\left(\hat{\imath}\,X\frac{\partial B}{\partial x}\right).A$$

$=curl\vec{A}.\vec{B} - curl\vec{B}.\vec{A}$

From the identity,

$\vec{a}.(b\,X\,\vec{c}) = (\vec{a}\,X\,b).\vec{c} = -\vec{a}.(c\,X\,\vec{b})$

25.If $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ and $|\vec{r}| = r$ find $grad\left(div\frac{\vec{r}}{r}\right)$

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2} \qquad r^2 = x^2 + y^2 + z^2$$

$\frac{\partial r}{\partial x} = \frac{x}{r} \qquad \frac{\partial r}{\partial y} = \frac{y}{r} \qquad \frac{\partial r}{\partial z} = \frac{z}{r}$

$$div\frac{\vec{r}}{r} = \frac{d(r^{-1}x)}{dx} + \frac{d(r^{-1}y)}{dy} + \frac{d(r^{-1}z)}{dz}$$

$$r^{-1} + x(-1)r^{-2}\frac{\partial r}{\partial x} + y(-1)r^{-2}\frac{\partial r}{\partial y} + z(-1)r^{-2}\frac{\partial r}{\partial z}$$

$$= 3r^{-1} - r^{-3(x^2+y^2+z^2)} = 3r^{-1} - r^{-1} = 2r^{-1}$$

$$grad\left(div\frac{\vec{r}}{r}\right) = \frac{\partial}{\partial x}(2r^{-1})\hat{\imath} + \frac{\partial}{\partial y}(2r^{-1})\hat{\jmath} + \frac{\partial}{\partial z}(2r^{-1})\hat{k}$$

$$-2r^{-2}\frac{\partial r}{\partial x}\hat{\imath} - 2r^{-2}\frac{\partial r}{\partial y}\hat{\jmath} - 2r^{-2}\frac{\partial r}{\partial z}\hat{k}$$

26.Prove that $\nabla X(\nabla X\vec{F}) = \nabla(\nabla.\vec{F}) - \nabla\text{^}2\vec{F}$ where

$$\vec{F} = x^2y\hat{\imath} + zx\hat{\jmath} - 3yz^2\hat{k}$$

$$curl\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & zx & -3yz^2 \end{vmatrix}$$

$$= \hat{\imath}[-3z^2 - x] - \hat{\jmath}[0 - 0] + \hat{k}[z - x^2]$$

$$curl\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -3z^2 - x & 0 & z - x^2 \end{vmatrix}$$

$$= \hat{\imath}[0 - 0] - \hat{\jmath}[-2x + 6z] + \hat{k}[0 - 0]$$

$0\hat{\imath} + \hat{\jmath}(-2x + 6z) + 0\hat{k}$------------------------- 1.

$$\nabla\vec{F} = 2xy - 6yz$$

$$\nabla(\nabla.\vec{F}) = 2y\hat{\imath} + (2x - 6z)\hat{\jmath} - 6y\hat{k}$$

$$\nabla\text{^}2\,\vec{F} = \frac{\partial^2 F}{\partial x^2}\hat{\imath} + \frac{\partial^2 F}{\partial y^2}\hat{\jmath} + \frac{\partial^2 F}{\partial z^2}\hat{k} = 2y\hat{\imath} + 0\hat{\jmath} - 6y\hat{k}$$

$$\nabla(\nabla.\vec{F}) - \nabla\text{^}2\vec{F} = 2y\hat{\imath} + (2x - 6z)\hat{\jmath} - 6y\hat{k} - 2y\hat{\imath} + 6y\hat{k}$$

$= (2x - 6z)\hat{\jmath}$-------------------------------------- 2.

From 1. & 2. ,

L.H.S = R.H.S

27.Prove that $(\vec{F} X \nabla) X \vec{r} = -2\vec{F}$ where $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$$(\vec{F} X \nabla) X \vec{r} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ F_1 & F_2 & F_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}$$

$$\hat{\imath}\left[F_2 \frac{\partial}{\partial z} - F_3 \frac{\partial}{\partial y}\right] - \hat{\jmath}\left[F_1 \frac{\partial}{\partial z} - F_3 \frac{\partial}{\partial x}\right] + \hat{\jmath}\left[F_1 \frac{\partial}{\partial y} - F_2 \frac{\partial}{\partial x}\right]$$

$$(\vec{F} X \nabla) X \vec{r} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ F_2 \frac{\partial}{\partial z} - F_3 \frac{\partial}{\partial y} & F_1 \frac{\partial}{\partial z} - F_3 \frac{\partial}{\partial x} & F_1 \frac{\partial}{\partial y} - F_2 \frac{\partial}{\partial x} \\ x & y & z \end{vmatrix}$$

$$= \Sigma\hat{\imath}\left[F_2 \frac{\partial z}{\partial x} - F_1 \frac{\partial z}{\partial z}\right] - \left[F_1 \frac{\partial y}{\partial y} - F_3 \frac{\partial y}{\partial z}\right]$$

$$= \Sigma\hat{\imath}[-F_1 - F_1] = -2F_1\hat{\imath}$$

$$= -2F_1\hat{\imath} - 2F_2\hat{\jmath} - 2F_3\hat{k}$$

$$= -2[F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}]$$

$$= -2\vec{F}$$

28.If $\vec{V} = \vec{W} X \vec{R}$ prove that $curl\ \vec{V} = 2\vec{W}$ where $\vec{W}$ is a constant vector

Let $\vec{W} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

$$\vec{V} = \vec{W} X \vec{R} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$\vec{V} = \hat{\imath}[a_2 z - a_3 y] - \hat{\jmath}[a_1 z - a_3 x] + \hat{k}[a_1 y - a_2 x]$$

$$curl\,\vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_2 z - a_3 y & a_1 z - a_3 x & a_1 y - a_2 x \end{vmatrix}$$

$$curl\,\vec{v} = \hat{\imath}[a_1 + a_1] - \hat{\jmath}[-a_2 - a_2] + \hat{k}[a_3 + a_3]$$

$$= 2a_1\hat{\imath} + 2a_2\hat{\jmath} + 2a_3\hat{k}$$

$$= 2[a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}]$$

$$curl\,\vec{v} = -2\vec{v}$$

29.If $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ and $|\vec{r}| = r$ then prove that $\frac{\vec{r}}{r^3}$ is solenoidal

$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$ $\qquad r^2 = x^2 + y^2 + z^2$

$\frac{\partial r}{\partial x} = \frac{x}{r} \qquad \frac{\partial r}{\partial y} = \frac{y}{r} \qquad \frac{\partial r}{\partial z} = \frac{z}{r}$

$$div\frac{\vec{r}}{r^3} = div\left(xr^{-3}\hat{\imath} + yr^{-3}\hat{\jmath} + zr^{-3}\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(xr^{-3}) + \frac{\partial}{\partial y}(yr^{-3}) + \frac{\partial}{\partial z}(zr^{-3})$$

$$div\frac{\vec{r}}{r^3} = r^{-3}3xr^{-4}\frac{\partial r}{\partial x} + r^{-3}3yr^{-4}\frac{\partial r}{\partial y} - r^{-3}3zr^{-4}\frac{\partial r}{\partial z}$$

$$= 3r^{-3} - 3r^{-5}(x^2 + y^2 + z^2)$$

$$= 3r^{-3} - 3r^{-3} = 0$$

30.$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ and $|\vec{r}| = r$ then prove that $div(r^3\vec{r})$

$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$ $\qquad r^2 = x^2 + y^2 + z^2$

$\frac{\partial r}{\partial x} = \frac{x}{r} \qquad \frac{\partial r}{\partial y} = \frac{y}{r} \qquad \frac{\partial r}{\partial z} = \frac{z}{r}$

$$div(r^3\vec{r}) = div\left(xr^3\hat{\imath} + yr^3\hat{\jmath} + zr^3\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(xr^3) + \frac{\partial}{\partial y}(yr^3) + \frac{\partial}{\partial z}(zr^3)$$

$$= r^{-3} + x3r^2\frac{\partial r}{\partial x} + r^3 + 3yr^2\frac{\partial r}{\partial y} + r^3 + 3zr^2\frac{\partial r}{\partial z}$$

$$= 3r^3 + 3r(x^2 + y^2 + z^2)$$

$$= 3r^3 + 3r^3 = 6r^3$$

31.Find $\nabla^2 r^n$ where $r = |x\hat{\imath} + y\hat{\imath} + z\hat{k}|$ and further show that it is equal zero if $n = -1$

$$\nabla^2 r^n = \frac{\partial^2}{\partial x^2}(r^n) + \frac{\partial^2}{\partial y^2}(r^n) + \frac{\partial^2}{\partial z^2}(r^n)$$

$\frac{\partial^2}{\partial x^2}(r^n) = nr^{n-1}$ $\qquad\qquad$ $\frac{\partial r}{\partial x} = nxr^{n-2}$

$$\frac{\partial^2}{\partial x^2}(r^n) = n[r^{n-1} + x(n-2)r^{n-3}\frac{\partial r}{\partial x}]$$

$$r = |x\hat{\imath} + y\hat{\jmath} + z\hat{k}|$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$r^2 = x^2 + y^2 + z^2$$

$\frac{\partial r}{\partial x} = \frac{x}{r}$ $\qquad$ $\frac{\partial r}{\partial y} = \frac{y}{r}$ $\qquad$ $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$= n[r^{n-2}.1 + x^2(n-2)r^{n-4}]$$

$$= nr^{n-2} + n(n-2)r^{n-4}x^2$$

$$\frac{\partial^2}{\partial y^2}(r^n) = nr^{n-2} + n(n-2)r^{n-4}y^2$$

$$\frac{\partial^2}{\partial z^2}(r^n) = nr^{n-2} + n(n-2)r^{n-4}z^2$$

$$\nabla^2 r^n = 3nr^{n-2} + n(n-2)r^{n-4}(x^2+y^2+z^2)$$

$$= 3nr^{n-2} + n(n-2)r^{n-2}$$

$$= nr^{n-2}[3+n-2]$$

$$= nr^{n-2}[n+1] = 0$$

$$if\ \ n = -1$$

32.If $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ then show that 1. grad $\frac{1}{|\vec{r}|} = \frac{-\vec{r}}{|\vec{r}|^3}$

2.$grad|\vec{r}|^3 = 3|\vec{r}|\vec{r}$

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

$$|\vec{r}| = \sqrt{x^2+y^2+z^2} = (x^2+y^2+z^2)\text{^}1/2 \Rightarrow$$

$$|\vec{r}|^2 = (x^2+y^2+z^2)$$

$$\frac{1}{|\vec{r}|} = (x^2+y^2+z^2)\text{^}-1/2$$

$$grad\frac{1}{|\vec{r}|} = \frac{\partial}{\partial x}\frac{1}{|\vec{r}|}\hat{\imath} + \frac{\partial}{\partial y}\frac{1}{|\vec{r}|}\hat{\jmath} + \frac{\partial}{\partial z}\frac{1}{|\vec{r}|}\hat{k}$$

$$\frac{\partial}{\partial x}\frac{1}{|\vec{r}|} = -\frac{1}{2}[x^2+y^2+z^2]^{-\frac{1}{2}}2x = -x[x^2+y^2+z^2]^{-\frac{3}{2}}$$

$$= -x[|\vec{r}|^2]^{-\frac{3}{2}} = -x|\vec{r}|^3$$

Similarly, $\frac{\partial}{\partial y}\frac{1}{|\vec{r}|} = -y|\vec{r}|^3$ $\frac{\partial}{\partial z}\frac{1}{|\vec{r}|} = -z|\vec{r}|^3$

$$grad\ \frac{1}{|\vec{r}|} = -x|\vec{r}|^{-3}\hat{\imath} - y|\vec{r}|^{-3}\hat{\jmath} - z|\vec{r}|^{-3}\hat{k}$$

$$= -|\vec{r}|^{-3}[x\hat{\imath} + y\hat{\jmath} + z\hat{k}] = -\frac{\vec{r}}{|\vec{r}|^{-3}}$$

$$grad|\vec{r}|^3 = \frac{\partial}{\partial x}|\vec{r}|\hat{\imath} + \frac{\partial}{\partial y}|\vec{r}|^3\hat{\jmath} + \frac{\partial}{\partial z}|\vec{r}|^3\hat{k}$$

$$|\vec{r}|^3 = [x^2 + y^2 + z^2]^{\frac{3}{2}}$$

$$\frac{\partial}{\partial x}|\vec{r}|^3 = \frac{3}{2}(x^2 + y^2 + z^2)^{\frac{1}{2}}2x = 3x(x^2 + y^2 + z^2)^{\frac{1}{2}}$$

$$= 3x|\vec{r}|$$

Similarly, $\frac{\partial}{\partial y}|\vec{r}|^3 = 3y|\vec{r}|\frac{\partial}{\partial z}|\vec{r}|^3 = 3z|\vec{r}|$

$$grad|\vec{r}|^3 = 3x|\vec{r}|\hat{\imath} + 3y|\vec{r}|\hat{\jmath} + 3z|\vec{r}|\hat{k}$$

$$grad|\vec{r}|^3 = 3|\vec{r}|\left[x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right] = 3|\vec{r}|\vec{r}$$

33.If $\vec{A}$ and $\vec{B}$ are two irrotational vector function, then prove that $\vec{A}X\vec{B}$ is solenoidal

$$div\left(\vec{A}X\vec{B}\right) = \vec{B}.curl\vec{A} - \vec{A}.curl\vec{B}$$

$\vec{A}$ and $\vec{B}$ are irrotational, hence $curl\vec{A} = curl\vec{B} = 0$

Then $div\left(\vec{A}X\vec{B}\right) = 0$

Hence $(\vec{A}X\vec{B})$ is solenoidal

34.State the condition under which a vector function is irrotational and when it is solenoidal, prove that

$\vec{F} = (siny + z)\hat{\imath} + (xcosy - z)\hat{\jmath} + (x - y)\hat{k}$ is irrotational

$\vec{G} = (x + 3y)\hat{\imath} + (y - 2z)\hat{\jmath} + (x - 2z)\hat{k}$ is solenoidal

The function $\vec{F}$ is irrotational if curl$\vec{F}=0$ and is solenoidal if $\mathrm{Div}\vec{F}=0$

$$\mathrm{curl}\vec{F}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \sin y+z & x\cos y-z & x-y\end{vmatrix}$$

$$=\hat{\imath}[-1+1]-\hat{\jmath}[1-1]+\hat{k}[\cos y-\cos y]=0$$

$$=0\ is\ irrotational$$

$$\mathrm{Div}\vec{F}=\frac{\partial}{\partial x}(x+3y)+\frac{\partial}{\partial y}(y-2z)+\frac{\partial}{\partial z}(x-2z)$$

$$=1+1-2$$

35.Find curl$\vec{F}$ where $\vec{F}=\frac{x}{r}\hat{\imath}+\frac{y}{r}\hat{\jmath}+\frac{z}{r}\hat{k}$ and

$r=|x\hat{\imath}+y\hat{\jmath}+z\hat{k}|$

$$r=|x\hat{\imath}+y\hat{\jmath}+z\hat{k}|$$

$$r=\sqrt{x^2+y^2+z^2}$$

$$r^2=x^2+y^2+z^2$$

$$\frac{\partial r}{\partial x}=\frac{x}{r}\qquad \frac{\partial r}{\partial y}=\frac{y}{r}\qquad \frac{\partial r}{\partial z}=\frac{z}{r}$$

$$\mathrm{curl}\vec{F}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{x}{r} & \frac{y}{r} & \frac{z}{r}\end{vmatrix}$$

$$=\hat{\imath}\left[\frac{-z\frac{\partial r}{\partial y}}{r^2}+\frac{y\frac{\partial r}{\partial z}}{r^2}\right]-\hat{\jmath}\left[\frac{-z\frac{\partial r}{\partial x}}{r^2}+\frac{y\frac{\partial r}{\partial z}}{r^2}\right]+\hat{k}\left[\frac{-z\frac{\partial r}{\partial x}}{r^2}+\frac{y\frac{\partial r}{\partial y}}{r^2}\right]$$

$$=\hat{\imath}\left[-\frac{zy}{r^3}+\frac{zy}{r^3}\right]-\hat{\jmath}\left[-\frac{zx}{r^3}+\frac{zx}{r^3}\right]+\hat{k}\left[-\frac{yx}{r^3}+\frac{xy}{r^3}\right]=0$$

36.Prove that $div(\vec{A}+\vec{B}) = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)(A + B)$

$$= \hat{\imath}.\frac{\partial}{\partial x}[A+B] + \hat{\jmath}.\ \frac{\partial}{\partial y}[A+B] + \hat{k}.\ \frac{\partial}{\partial z}[\vec{A}+\vec{B}]$$

$$= \hat{\imath}.\left[\frac{\partial A}{\partial x} + \frac{\partial B}{\partial x}\right] + \hat{\jmath}.\left[\frac{\partial A}{\partial y} + \frac{\partial B}{\partial y}\right] + \hat{k}.\left[\frac{\partial A}{\partial z} + \frac{\partial A}{\partial z}\right]$$

$$\left[\hat{\imath}.\frac{\partial A}{\partial x} + \hat{\jmath}.\frac{\partial A}{\partial y} + \hat{k}.\frac{\partial A}{\partial z}\right] + \left[\hat{\imath}.\frac{\partial B}{\partial x} + \hat{\jmath}.\frac{\partial B}{\partial y} + \hat{k}.\frac{\partial B}{\partial z}\right]$$

$$= div\vec{A} + div\vec{B}$$

37.With usual notation, prove that $\nabla^2 f(r) = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial r}{\partial f}$

where $r^2 = x^2 + y^2 + z^2$

Let $\phi = f(r)$ Diff. w.r.t. x

$$\frac{\partial \phi}{\partial x} = f'(\mathrm{r})\frac{\partial \mathrm{r}}{\partial \mathrm{x}} = f'(r)\frac{x}{r}$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{x}{r}f''(r)\frac{\partial r}{\partial x} + f'(r)[\frac{r - x\frac{\partial r}{\partial x}}{r^2}]$$

$$= \frac{x^2}{r^3}f''(r) + f'(r)[\frac{1}{r} - \frac{x^2}{r^3}]$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{x^2}{r^2}f''(r) + \frac{f'(r)}{r} - \frac{x^2 f'(r)}{r^3}$$

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{y^2}{r^2}f''(r) + \frac{f'(r)}{r} - \frac{y^2 f'(r)}{r^3}$$

$$\frac{\partial^2 \phi}{\partial z^2} = \frac{z^2}{r^2}f''(r) + \frac{f'(r)}{r} - \frac{z^2 f'(r)}{r^3}$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$= \frac{f''(r)}{r^2}(x^2 + y^2 + z^2) + \frac{3f'(r)}{r} - \frac{f'(r)}{r^3}(x^2 + y^2 + z^2)$$

$$= f(\mathrm{r}) + \frac{3f'(r)}{r} - \frac{f'(r)}{r}$$

$$\nabla^2 f(r) = f(\mathrm{r}) + \frac{2f'(r)}{r}$$