Analytical Solid Geometry

- **Distance formula(without proof)**
- **Division Formula**
- **Direction cosines**
- **Direction ratios**
- **Planes**
- **Straight lines**

Books

- **Higher Engineering Mathematics**

  By B S Grewal

- **Higher Engineering Mathematics**

  By H K Das

Coordinates and Direction cosines

- One position of a point in a plane is usually specified by two real numbers, x and y depend upon the chosen system of reference.
- But the position of a point in space is specified by three numbers x, y and z.

Here the 'P' is located by three Cartesian coordinates x, y and z, the axes Ox, Oy and Oz are mutually perpendicular to each other.

Types of Coordinate System

- **Rectangular Coordinate system:**

  Here ‘P’ is located by the coordinates (x,y) and z which are called Rectangular coordinates.

- **Curvilinear Coordinate System**

  There are two type of curvilinear coordinate system,

  (i) Spherical polar coordinate system (ρ,θ,φ)

  $x = \rho \sin\theta\cos\varphi \quad = \rho \sin\theta \sin\varphi \quad z = \rho\cos\varphi$ y

  (ii) Cylindrical Polar Coordinate system (ρ,θ,Z)

  $x = \rho \cos\theta \quad y= \rho \sin\theta \quad z = z$

Distance between two points

- Let $P(x_1,y_1,z_1)$ and Q (x_2,y_2,z_2) be the two given points,
  PQ= $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Ratio formula (section formula)

- Let P(x,y,z) and Q(x,y,z) be the two given points, PQ divides internally in the ration $m_1 : m_2$, let R(x,y,z) be the dividing point

$$x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}, y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}, \quad z = \frac{m_1z_2 + m_2z_1}{m_1 + m_2}$$

- If R is the midpoint, $m_1 = m_2$

  $R(x, y, z) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$

- If , $m_1 : m_2 = \lambda:1$

  $R(x, y, z) = \left(\frac{\lambda x_1 + x_2}{2}, \frac{\lambda y_1 + y_2}{2}, \frac{\lambda z_1 + z_2}{2}\right)$

- If R(x,y,z) divides externally $m_1 = m_2,$

  $R(x, y, z) = \left(\frac{m_1x_1 - m_2x_2}{2}, \frac{m_1y_1 - m_2y_2}{2}, \frac{m_1z_1 - m_2z_2}{2}\right)$

Centroid of a triangle whose vertices are given

Let A, B,C be the vertices of the triangle ABC. If D be the mid point of the BC then , then G – Centroid, which divides AD in the ratio 2:1

$$G(x,y,z) = \left(\frac{x_1 + x_2 + x_3}{2}, \frac{y_1 + y_2 + y_3}{2}, \frac{z_1 + z_2 + z_3}{2}\right)$$

Centroid of a tetrahedron whose vertices are given

A tetrahedron is a solid bounded by four triangular faces at six edges and has four vertices A,B,C,D. Let G be the centroid of the tetrahedron ABCD divides AG in 3:1 then,

$$G(x,y,z) = \left(\frac{x_1 + x_2 + x_3 + x_4}{2}, \frac{y_1 + y_2 + y_3 + y_4}{2}, \frac{z_1 + z_2 + z_3 + z_4}{2}\right)$$

OR $$G(x,y,z) = \left(\frac{\sum x}{4}, \frac{\sum y}{4}, \frac{\sum z}{4}\right)$$

Direction cosines of a line (l,m,n)

The cosines of the angles made by any line with the positive directions of the coordinate axes are called the Direction cosines of a line.

Let α, β, γ be the angles with x, y, z axes then,

Then ; $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$ are direction cosines

Coordinates of a point on a line whose direction cosines are given,

$P(x,y,z) = (lr, mr, nr)$

Relation between Direction Cosines, l,m,n or a line

$l^2 + m^2 + n^2 = 1$; $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$; $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

Direction Ratios (proportional Direction cosines)

A set of three numbers a,b,c to which D.C's of a line are proportional are called Direction Ratios.

Direction cosines from the Direction ratios

Let a, b, c be the D.R's whose D.C's are l, m, n then,

$$l = \frac{a}{\sqrt{a^2 + b^2 + C^2}} \; ; \; m = \frac{b}{\sqrt{a^2 + b^2 + C^2}} \; ; \; n = \frac{c}{\sqrt{a^2 + b^2 + C^2}}$$

D.C's of line joining (x_1, y_1, z_1) and (x_2, y_2, z_2)

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be the given points. PQ makes α,β,γ with x, y, z axes respectively, then,

$l = \frac{x_2 - x_1}{PQ}$; $m = \frac{y_2 - y_1}{PQ}$; $n = \frac{z_2 - z_1}{PQ}$

Angle between two lines having give D.C's.

Let AB and CD be two lines whose D.C's are l_1, m_1, n_1 and l_2, m_2, n_2 respectively, θ be the angle. Through 'OP' draw OP and OQ parallel to AB and CD.

Therefore, $\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$

Note: if cosθ is –ve, θ between the two lines is oblique angle and acute at (180- θ)

Expression for sinθ and tanθ

Expression for sinθ is given by

$$\sin\theta = \sqrt{\sum (l_1 m_2 - l_2 m_1)^2}$$

Expression for tanθ is given by

$$\tan\theta = \frac{\sqrt{\sum (l_1 m_2 - l_2 m_1)^2}}{l1\, l2 + m1\, m2 + n1 n2}$$

Condition for three points to be collinear

Let Let $A(x_1,y_1,z_1)$,B (x_2,y_2,z_2) and $C(x_3,y_3,z_3)$ be three given points which are collinear if the D.C's of AB and BC are equal or their D.R's are proportional .

Therefore required condition for co linearity is

$$\frac{x_2 - x_1}{x_3 - x_2} = \frac{y_2 - y_1}{y_3 - y_1} = \frac{z_2 - z_1}{z_3 - z_1}$$

Condition for three concurrent lines to be coplanar

Let OP, OQ, OR be three concurrent lines with l_1, m_1, n_1 and l_2, m_2, n_2 and l_3, m_3, n_3 as D.C's respectively. If these are coplanar if there exists a straight line perpendicular to OA then,

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0$$

Projection of the line joining two points on line

Let AB be the given line with D.C's l,m,n. Let $P(x_1,y_1,z_1)$ and R (x_2,y_2,z_2) be the given points. Let θ be the angle between AB and PR, then

Projection of PR on AB = $l(x_2-x_1) + m(y_2 - y_1) + n(z_2 - z_1)$

Problems:

1. Find the direction cosines of a line whose direction ratios are 3,-4, 5.

   Soln: Given D.R's are 3,-4, 5 $\Leftrightarrow$ a, b, c

    We have the relations between D.C's and D.R's as,

   $$l = \frac{a}{\sqrt{a^2+b^2+C^2}} \; ; m = \frac{b}{\sqrt{a^2+b^2+C^2}} \; ; \; n = \frac{c}{\sqrt{a^2+b^2+C^2}}$$

   $$\mathbf{l} = \frac{3}{\sqrt{3^2+(-4)^2+5^2}} = \frac{\mathbf{3}}{\sqrt{\mathbf{50}}} = \frac{\mathbf{3}}{\mathbf{5\sqrt{2}}}, \mathbf{m} = \frac{\mathbf{-4}}{\mathbf{5\sqrt{2}}}, \; \mathbf{n} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}$$

   which are also called as actual D.C's

2. Obtain the D.C's of a line equally inclined to the axes.

   Soln. For equally inclined, $\alpha = \beta = \gamma$ $\cos\alpha = \cos\beta = \cos\gamma$

    $l = m = n$, $l^2 + m^2 + n^2 = 1$, $l^2 + l^2 + l^2 = 1$ $l = \frac{1}{\sqrt{3}}$

    D.C's are l, m , n $\pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}$

3. Find the coordinates of the point which divides the join the (1,-2, 3) and (3, 4,-5) in the ration 2:3 internally and in the 2:3 externally.

   Soln. From the section formula, we have internal division,

   $$x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}, y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}, \; z = \frac{m_1z_2 + m_2z_1}{m_1 + m_2}$$

   $$\mathbf{x} = \frac{2 \text{ x } 3 + 3 \text{ x } 1}{2 + 3} = \frac{\mathbf{9}}{\mathbf{5}}; \mathbf{y} = \frac{\mathbf{2}}{\mathbf{5}}; \; \mathbf{z} = \frac{\mathbf{-1}}{\mathbf{5}}$$

   For external division,

   $$x = \frac{m_1x_2 - m_2x_1}{m_1 - m_2}, y = \frac{m_1y_2 - m_2y_1}{m_1 - m_2}, \; z = \frac{m_1z_2 - m_2z_1}{m_1 - m_2}$$

   $$\mathbf{x} = \frac{2 \text{ x } 3 - 3 \text{ x } 1}{2 - 3} = \mathbf{-3}; \; \mathbf{y} = \mathbf{-14}; \; \mathbf{z} = \mathbf{19}$$

4. Find the coordinates of the midpoint of the points (1,2,3,) and (3,-6,7).

Soln. For midpoint, m = n ;

$$x = \frac{x_1 + x_2}{2}, \qquad y = \frac{y_1 + y_2}{2}, \qquad z = \frac{z_1 + z_2}{2}$$

x = 2 , y = -2 , z = 5

5. Find the ratio in which the line joining the points (3, 1, 5), (-2, 4,-3) is divided by the xy-plane and also the coordinates of the pint of intersection.

Soln. General coordinates of any points on the line joining the given two points,

$$\frac{\lambda x_1 + x_2}{\lambda + 1}, \frac{\lambda y_1 + y_2}{\lambda + 1}, \frac{\lambda z_1 + z_2}{\lambda + 1}$$

□ $P(x, y, z) = \frac{-2\lambda + 3}{\lambda + 1}, \frac{4\lambda + 1}{\lambda + 1}, \frac{-3\lambda + 5}{\lambda + 1}$

Since the line is divided by xy-plane, z = 0, then,

$$\frac{-3\lambda + 5}{\lambda + 1} = 0 \Rightarrow \lambda = \frac{5}{3} \Rightarrow \mathbf{m : n = 5 : 3}$$

□ $\mathbf{P(x, y, z)} = \left(\frac{\mathbf{-1}}{\mathbf{8}}, \frac{\mathbf{23}}{\mathbf{8}}, \mathbf{0}\right)$

6. The D.C's are l,m,n are connected by the relational l + m + n = 0, 2lm+2ln-mn = 0. Find them.

Soln. Given l + m + n = 0, 2lm+2ln-mn = 0.

We solve the equations, □ $n = -l - m$.

2lm+2l(- l - m) – m(- l - m) = 0.

Solving we get, $\mathbf{l_1} = \frac{1}{\sqrt{6}}$; $\mathbf{m_1} = \frac{1}{\sqrt{6}}$; $\mathbf{n_1} = \frac{-2}{\sqrt{6}}$ **and** $\mathbf{l_2} = \frac{1}{\sqrt{6}}$; $\mathbf{m_2} = \frac{-2}{\sqrt{6}}$; $\mathbf{n_2} = \frac{1}{\sqrt{6}}$

7. Prove that the three points P, Q, R whose coordinates are (3, 2,-, 4), (5,4,-6) and (9,8,-10) respectively are collinear, find the ratio in which the point Q dividing PR.

Soln. For co linearity, $\frac{x_2 - x_1}{x_3 - x_2} = \frac{y_2 - y_1}{y_3 - y_1} = \frac{z_2 - z_1}{z_3 - z_1}$

Let Q divides PR in the ratio, λ : 1, □ $\left(\frac{9\lambda+3}{\lambda+1}, \frac{8\lambda+2}{\lambda+1}, \frac{-10\lambda-4}{\lambda+1}\right)$,

Comparing with Q (any point)

Let $\frac{9\lambda+3}{\lambda+1} = 5$ □ $\lambda = \frac{1}{2}$, □ $\boldsymbol{\lambda} : \mathbf{1} = \frac{1}{2} : 1$ □ **1 □ 2**

8. A line makes angles α, β, γ and δ with the four diagonals of a cube, show that (1) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{4}{3}$ (2) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = \frac{8}{3}$

Soln. Length of the cube be 'a' units AA', BB', CC' , OP be the diagonals 0(0,0,0), A(a,0,0), A'(0,a,a), B(0,a,0), B'(a,0,a), C(0,0,a), C'(a,a,0), P(a, a, a) are the coordinates of the vertices,

D.R's of AA^1, BB^1, CC^1, OP are (-a,a,a), (a,-a,a),(a,a,-a) and (a,a,a)

Therefore, D.C's of $AA^1 = -\frac{a}{\sqrt{a^2+b^2+c^2}} = -\frac{l}{\sqrt{3}} = l$, $m = \frac{1}{\sqrt{3}}, n = \frac{1}{\sqrt{3}}$

D.C's of $BB^1 = \frac{l}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$; $CC^1 = \frac{l}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$ and OP are $\frac{l}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

If l, m, n are the C.C's of the given line we have $l^2 + m^2+n^2 = 1$

$$\cos\alpha = \frac{-l+m+n}{\sqrt{3}}, \cos\beta = \frac{l-m+n}{\sqrt{3}}, \cos\gamma = \frac{l+m-n}{\sqrt{3}}, \cos\delta = \frac{l+m+n}{\sqrt{3}}$$

$$\mathbf{\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{1}{3}(4l^2 + 4m^2 + 4n^2) = \frac{4}{3}}$$

$$\mathbf{\sin^2\alpha + \sin^2\beta + \sin^2\gamma + \sin^2\delta = \frac{8}{3}}$$

9. Find the coordinates of the point which divides the line joining the points (2,-3,4) and (0,-1,3) in the ratio 3:2 and also find the midpoint.

Soln. (x, y, z) ≡ (2,-3,4) , (x, y, z) ≡ (0,-1,3) and $m_1 : m_2 \equiv 3:2$

$$P(x,y,z) = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}, \frac{m_1z_2 + m_2z_1}{m_1 + m_2}$$

$$\mathbf{P(x,y,z) = \frac{4}{5}, \frac{-9}{5}, \frac{17}{5}}$$

Midpoint is $\mathbf{R(x,y,z)} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$ = **(1,-2, 7/2)**

10. Find the perimeter (length) of the triangle whose vertices are (1,1,1) , (1,-2,1) and (-1,0,-2) and also find the coordinates of the Centroid of the triangle.

Soln. Let A,B,C be the vertices of a triangle, then, $AB = 3, BC = \sqrt{17}, AC = \sqrt{14}$

Perimeter = AB+BC+AC = $\mathbf{3 + \sqrt{17} + \sqrt{14}}$.

Centroid = $\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3} = \mathbf{\frac{1}{3}, \frac{-1}{3}, 0}$

11. Find the coordinates of the foot the perpendicular from (1,2,3) on the line joining (1,3,7) and (4,3,10).

Soln. Let 'L' be the foot of perpendicular AL, which divides BC in the ratio λ : 1

$$= \left(\frac{4\lambda+1}{\lambda+1}, \frac{3\lambda+3}{\lambda+1}, \frac{10\lambda+7}{\lambda+1}\right),$$ Also D.R's of the BC and AL are ,

D.R's of BC; 4 - 1,3 - 3,10 – 7 = (3,0,3)

D.R's of AL ; $\left(\frac{4\lambda+1}{\lambda+1} - 1, \frac{3\lambda+3}{\lambda+1} - 2, \frac{10\lambda+7}{\lambda+1} - 3\right)$ since AL is perpendicular to BC

Cos90 = 0 →BC, QL are the lines with D.R's as magnitudes

$$(BC)(AL) = \left(\left(\frac{4\lambda+1}{\lambda+1} - 1\right)3, \left(\frac{3\lambda+3}{\lambda+1} - 2\right)0, \left(\frac{10\lambda+7}{\lambda+1} - 3\right)3\right)$$

λ = -2/5, L = (-1, 3, 5)

12. Show that pair of lines whose direction cosines are given by the equations 2l – m + 2n = 0, mn + nl + lm = 0 are perpendicular.

Soln. Given 2l – m + 2n = 0 □ m = 2l + 2n , substituting in

mn + nl + lm = 0 □ (2l + 2n) n +nl + l(2l + 2n) = 0, solving we get,

□ $2l^2 + 5ln + 2n^2 = 0$ which is quadratic equation

When,

l = -n/2	l = -2n
m = -n/2 x 2 +2n = **n**	m = -4n + 2n = **-2n**

The D.C's are ,l = -n/2, m = n , n = n are one set of D.C's

and $l_1 = -2n$, $m_1 = -2n$, $n_1 = n$ are second set of D.C's

since the two lines are perpendicular then, $ll_1 + mm_1 + nn_1 = 0$ then,

$n^2 - 2n^2 + n^2 = 0$ is satisfied.

13. Prove that the lines whose D.C's are given the relations al + bm + cn = 0 and mn +nl +lm = 0 are (i) perpendicular if 1/a +1/b+1/c = 0
(ii) parallel if $a^{1/2} + b^{1/2} + c^{1/2} = 0$

Soln. al +bm +cn = 0 □ n = -(al + bm) /c, substituting , divided by m^2 , we get,

$a(l/m)^2 + (c - a - b)(l/m) + b = 0$ which is quadratic in (l/m),

then solution is given by

$$\frac{l_1 l_2}{\frac{1}{a}} = \frac{m_1 m_2}{\frac{1}{b}} = \frac{n_1 n_2}{\frac{1}{c}} = k \;\Rightarrow\; l_1 l_2 = \frac{k}{a},\; m_1 m_2 = \frac{k}{b},\; n_1 n_2 = \frac{k}{c}$$

For perpendicular, $l_1 l_2 + m_1 m_2 + n_1 n_2 = k(1/a + 1/b + 1/c)$ iff, $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

$\Rightarrow$ **$1/a + 1/b + 1/c = 0$**

If the lines are parallel,

$l_1 = l_2, m_1 = m_2, n_1 = n_2$

we have the condition for parallel roots the discriminant

$\Delta = b^2 - 4ac$ here b = - (- a - b), hence, a = a , c = b

$b^2 - 4ac = (c - a - b)^2 - 4ab$ solving we get,

$a^{1/2} + b^{1/2} + c^{1/2} = 0$

14. A straight line is inclined to the axes y and z at angles 45^0 and 60^0. Find the inclination the x- axis.

Soln. Let α be the inclination to x – axis, then $\beta = 45^0$ and $\gamma = 60^0$

We have, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, substituting we get, **$\alpha = 60^0$**

15. Find the angle between the lines whose D.C's are (1,-2,3) and ((2,4,2).

Soln. $l_1, m_1, n_1 = (1,-2,3)$ and $l_2, m_2, n_2 = ((2,4,2)$, we have

$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$, Substituting we get **$\theta = 90^0$**

16. If the two lines have D.C's proportional (1,2,3) and (-2, 1,3) respectively. Find the D.C's of a line perpendicular to the both of the line.

Soln. D.C's of the given line are $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$ and $\frac{-2}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}$

Let l,m,n be the D.C's of a line perpendicular to between of them,

$\frac{l}{\sqrt{14}}, \frac{2m}{\sqrt{14}}, \frac{3n}{\sqrt{14}} = 0 \Rightarrow l + 2m + 3n = 0$ *and* $\frac{-2l}{\sqrt{14}}, \frac{m}{\sqrt{14}}, \frac{3n}{\sqrt{14}} = 0 \Rightarrow -2l + m + 3n = 0$

Solving by cross multiplication , we get $\frac{l}{\begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix}} = \frac{m}{\begin{vmatrix} 1 & 3 \\ -2 & 3 \end{vmatrix}} = \frac{n}{\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix}}$

(l, m, n) ≡ (3, -9, 5)

17. Find the projection of the line joining A(1,-2,2) and B(-1,2,0) on aline which makes an angle 30^0 with AB.

Soln. projection of line on AB = AB cos 30^0

$$\sqrt{(1+1)^2 + (-2-1)^2 + (2-0)^2} \quad = \frac{\sqrt{5}}{2}$$

18. Find the projection of the line joining the points A(1,2,3) and B(-1,1,0) on the line whose D.C's are (2,3,-1)

Soln. Projection of AB on the line with D.C's (2,3,-1)

$$= l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1) = 2(-1-1) + 3(1-2) \pm 1(0-3) = \mathbf{-4}$$

19. Set the points A(1,2,3), B(-1,3,4) and C(3,1,2) are collinear.

Soln. Condition for three points are collinear is $\frac{x_2 - x_1}{x_3 - x_2} = \frac{y_2 - y_1}{y_3 - y_1} = \frac{z_2 - z_1}{z_3 - z_1} = -\frac{\mathbf{1}}{\mathbf{2}}$

20. Set A(2,3,5), B(-1,5,1) and C(4,-3,2) form an isosceles right angled triangle.

Soln. $AB^2 = 49$, $BC^2 = 98$,$CA^2 = 49$ □ AB = CA, $BC^2 = AB^2 + CA^2$

Therefore A = 90^0 , triangle ABC is isosceles right angled triangle.

21. Find the coordinates of the foot of the perpendicular from A(0,9,6) on the line joining B(1,2,3) and C(7,-2,5).
22. Find the D.C's of the line which is perpendicular to the lines with D.C's proportional to (1,-2,-2) and (2,2,1)
23. Find the D.R's of a line perpendicular to the two lines whose direction rations are (-1, 2, 3) and (2, 3, 2).

Soln. Let L_1 and L_2 are the given lines with L_1 = (-1, 2, 3) = (a_1, b_1, c_1) and

L_2 = (2, 3, 2) = (a_2, b_2, c_2)

Let L_3 be the line perpendicular to L_1 and L_2 has the D.R.'s a, b, c. Also L3 is perpendicular to L_1 and L_2, then $\theta = 90^0$, cos90 = 0,

$aa_1 + bb_1 + cc_1 = 0$ = -a +2b + 3c = 0

$aa_2 + bb_2 + cc_2 = 0$ = 2a + 3b -2c = 0

by cross multiplication method, **a = -13, b = 4, c = -7**

24. Find the coordinates of the foot the perpendicular drawn from A(-3,-16,6) to the line joining B(4,-1,3), C(0,5,-2).
25. Find the angle between the lines whose D.C's are given by the equation 3l + m + 5n = 0, 6mn – 2nl +5lm = 0.

26. Find the D.Cs of the line which is perpendicular to the lines whose D.C's are proportional to (1,-2,-2), and ((0,2,1).
27. Set the lines whose D.C's are given by the relations, **$l + m + n = 0$** and **$al^2 + bm^2 + cn^2 = 0$** are perpendicular of **$a + b + c = 0$** and parallel of **$1/a + 1/b + 1/c = 0$.**
28. If A(1,4,2), B(-2,1,2), C(2,-3,4) be the points. Find the angles of triangle ABC. Find the D.R's of AB,AC, BC.
29.

PLANE

Defn : - A plane is surface such that straight line joinng any two points lies entirely in the surface. **OR** An equation which involves one or more of the current coordinates of a variable point in moving space is said to represent a surface which may be either plane or curved. **OR** A plane is a surface in which the straight line joining any two points on it lies wholly on it.

General equation of a plane:

It is of the form **$ax + by + cz + d = 0$.**

One point from of plane:

Let the general form is $ax + by + cz + d = 0$. Since it passes through (x,y,z), $ax_1 + by_1 + cz_1 + d = 0$, subtracting, **$a(x - x_1) + b(y - y_1) + c(z - z_1) + d = 0$** is called as "one point form of the plane"

Three point form of the plane through (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3).

Let the general form be $ax + by + cz + d = 0$ through (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3).

$ax_1 + by_1 + cz_1 + d = 0$

$ax_2 + by_2 + cz_2 + d = 0$

$ax_3 + by_3 + cz_3 + d = 0$,

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

Condition for four point to be coplanar

Let the general form be $ax + by + cz + d = 0$ through (x_1, y_1, z_1), (x_2, y_2, z_2) (x_3, y_3, z_3) and (x_4, y_4, z_4)

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

Intercept form equation of a plane having intercepts a, b, c on the axis

Let a, b, c be the intercepts with the equation of plane be

$\alpha x + \beta y + \gamma z + d = 0$ --------(1)

Plane passes through A(a, 0 ,0), B(0,b,0), C(0,0,c).

Through	A(a, 0 ,0),	$\alpha x + 0 + 0 + d = 0$,	$\alpha = -d/a$
	B(0,b,0) ,	$0 + \beta y + 0 + d = 0$,	$\beta = -d/b$
	C(0,0,c).	$0 + 0 + \gamma z + d = 0$,	$\gamma = -d/c$

Therefore equation (1) becomes

$(-d/a)\, x – (-d/b)\, y – (-d/c)\, z + d = 0$

$x/a + y/b + z/c = 1$ is the **intercept form**

Normal form of the plane having 'P' (perpendicular) from the origin and l,m,n as Direction Cosine's.

OL = OP = projection of OP on OL. Also OL = projection of the line joining P(x,y,z) on OL with D.C's l,m,n.

Therefore, $P = l(x-0) + m(y-0) + n(z-0)$

$P = lx + my + nz$ be the normal form. P is always positive.

Note : 1. An equation of the plane will be in normal form is

(Coeff. of x)^2 + (Coeff. of y)^2 + (Coeff, of z)^2 = 1.

2. **$x \cos\alpha + y \cos\beta + z \cos\gamma = P$**

3. Normal form of plane from General form,

$$\frac{ax}{\sqrt{a^2+b^2+C^2}} + \frac{by}{\sqrt{a^2+b^2+C^2}} + \frac{cz}{\sqrt{a^2+b^2+C^2}} = -\frac{d}{\sqrt{a^2+b^2+C^2}}$$

4. D.C's of the normal to the plane are proportional the coefficients of the x,y,z.

Plane through the intersection of the two planes

Let $ax_1 + by_1 + cz_1 + d_1 = 0$ and $ax_2 + by_2 + cz_2 + d_2 = 0$ be two planes, then equation of the plane is given by,

$$(ax_1 + by_1 + cz_1 + d_1) + \lambda(ax_2 + by_2 + cz_2 + d_2) = 0$$

Angle between two planes

Let the two planes be $ax_1 + by_1 + cz_1 + d_1 = 0$ and $ax_2 + by_2 + cz_2 + d_2 = 0$, hence, D.R's are

a_1, b_1, c_1 and a_2, b_2, c_2.

Let θ be the angle between two planes, then,

$$\cos\theta = \pm\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a^2 + b^2 + c^2}\ \sqrt{a^2 + b^2 + c^2}}$$

Note : 1. If the planes are perpendicular then $\theta = 90^0$, $\cos 90 = 0$

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

2. Two planes are parallel $\theta = 0^0$ or 180^0, $\cos 0 = 1$

$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ □ point of intersection is zero

3. Any plane parallel to the plane $ax + by + cz + d = 0$ is $\mathbf{ax + by + cz + k = 0}$, where k is to be evaluated.

Perpendicular distance of point from a plane.

If $ax + by + cz + d = 0$ is $\quad P(x, y, z) = \frac{ax_1 + by_1 + cz_1 + d}{\pm\sqrt{a^2 + b^2 + c^2}}$

Equations of the bisectors of the angles between two planes.

Let $ax_1 + by_1 + cz_1 + d_1 = 0$ and $ax_2 + by_2 + cz_2 + d_2 = 0$ be two planes, P(x,y,z) be any point at any one of the planes bisecting the angle between two given planes.

Perpendicular distance from P(x,y,z) from each plane is same.

$$P\,(x,y,z) = \frac{ax_1 + by_1 + cz_1 + d_1}{\pm\sqrt{a^2 + b^2 + C^2}} = \pm\frac{ax_2 + by_2 + cz_2 + d_2}{\pm\sqrt{a^2 + b^2 + C^2}}$$

Volume of a tetrahedron having given its vertices.

Let $A(x_1,y_1,z_1)$, $B(x_2,y_2,z_2)$, $C(x_3,y_3,z_3)$ and $D(x_4,y_4,z_4)$ be the vertices of tetrahedron ABCD, Volume is given by,

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

PROBLEMS

1.Find the equation of the plane passes through he points (0,1,1), (1,1,2) and (-1,2,-2).

Soln. Let the equation of the plane be is ax +by +cz +d = 0. If it passes through (0,1,1),

a(x-0) +b(y-1) +c(z-1) = 0 - (1)

If eq.(1) passing through (1,1,2) and (-1,2,-2).then, solving we get,

x - 2y –z + 3= 0

2.Find the equation of the plane which passes through the point (3,-3.1) and is

i) Parallel to the plane 2x +3y + 5z +6 = 0

ii) Normal to the line joining the points (3,2,-1) and (2,-1,5).

iii) Perpendicular to the planes 7x +y +2z = 6 and 3x +5y -6z = 8

Soln. Given plane is 2x +3y + 5z +6 = 0,

i) Any plane parallel to it is 2x +3y + 5z + k = 0, through (3,-3.1) is 6 – 9 + k = 0, **k = -2**

Hence required plane is **2x +3y + 5z - 2 = 0**

ii) Any plane through (3,-3.1) is , a(x-3) +b(y+3) + c(z-1) = 0.

D.C’s of the line joining (3,2,-1) and (2,-1,5) are proportional to

$(x_2-x_1),(y_2-y_1),(z_2-z_1)$ = (1,3,-6)

Line is normal to the plane, = a = 1, b = 2 ,c = -6

Therefore, 1(x-3) + 3(y+3) -6(z-1) = 0, = **x + 3y -6z +12 = 0**

iii) Any plane through (3,-3.1) is , a(x-3) +b(y+3) + c(z-1) = 0.

This is perpendicular to the planes 7x +y +2z = 6 and 3x +5y -6z = 8,

We know that $a_1a_2 + b_1b_2 + c_1c_2 = 0$, therefore, a = 1, b= -3, c = -2,

x - 3y -2z -10 = 0

3. Find the ratio in which the line joining of (4,-2,-3) and (-2,1,4) is divided by the plane 2x – 3y –z + 3 = 0.

Soln. Given eqn. is 2x – 3y –z + 3 = 0. Let A(4,-2,-3) and B(-2,1,4) be the given points.

Let P(x,y,z) be the point divides in λ:1 then, $P(x, y, z) = \left(\frac{-2\lambda+4}{\lambda+1}, \frac{\lambda-2}{\lambda+1}, \frac{4\lambda-3}{\lambda+1}\right)$

Substituting in plane we get λ = 5/2 = λ:1 = 5:2

4. Find the equation plane through the point (1,1,1) and through the intersection of the planes x + 2y +3z +4 = 0 and 4x +3y + 2z +1 = 0

5. Find the equation of the plane which passes through the line of cross-section of the planes 2x +y –z = 2 and x-y+2z = 3 and perpendicular to the plane x +y +z = 9.

6. Find the equation of the plane passing through the line of cross-section of the planes 2x - y +5z = -3 and 4x +2y-z+7 = 0 and parallel to the z-axis.

7.Find the intercepts made by the plane 2x +3y –z = -1 on the coordinate axes and also find the D.C’s of the normal to the plane.

8.Verify tha t the points (1,-1,0), (2,1,-1), (-1,3,1) and (-2,1,1) for coplanar and find the equation of the common plane.

9.Show that the Four points (0,4,3), (-1,-5,-3), (-2,-2,1) and (1,1,-1) are coplanar , find the equation of common plane.

10. Find the angle between the planes 2x - y + z = 6 and x+y+2z = 7.

11. Find the intercepts made by the plane 3x +4y –z = -6 on the coordinate axes and find the D.C’s of the normal to the plane.

12. Find the equation of the bisector of the angle between the planes 2x +y +2z -5 = 0 and 3x - 4y +1 = 0.

STRAIGHT LINE

STRAIGHT LINE

Definition: A plane cuts another plane in a line, therefore a straight line in a space is represented by two equation of the first degree in x, y, z

General form:

Let the points of cross-section of the two lines on the cross-section of two planes represents straight line i.e.

$ax_1 + by_1 + cz_1 + d_1 = 0$ and

$ax_2 + by_2 + cz_2 + d_2 = 0$

together represents straight line.

Symmetrical form of the equations of a straight line

AB is a line with D.C's l, m, n. P(x,y,z) is a point on the line AB, AP = r.

$RS = x - x_1 = AP \cos\alpha = rl$

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r$$

Therefore equations of a line AB are

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r$$

is called as **symmetrical form**.

Note:

Coordinates of any point on the line, we have

$$\frac{x - x_1}{l} = r \Rightarrow x = x_1 + lr\ , similarly\ y = y_1 + mr, z = z_1 + nr$$

$P(x, y, z) = (x_1 + lr, y_1 + mr, z_1 + nr)$

Equation of a line through (x,y,z) and having D.R's a,b,c.

Since the D.C's of a line are proportional to D.R's a,b,c. i.e., l,m,n ≡ a,b,c

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

Equation (1),i.e., symmetrical form, r will be distances from P(x,y,z) from $A(x_1,y_1,z_1)$ if and only if l,m,n are the actual D.C's.

Equation of the line passing through the two given points:

Let $A(x_1,y_1,z_1)$ and $B(x_2,y_2,z_2)$ be the two points.

Therefore D.R's are (x_2-x_1), (y_2-y_1), (z_2-z_1) ≡ a, b, c

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

INTERSECTION OF A LINE AND A PLANE

Let the lines $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$ and plane be ax +by +cz +d = 0.

Coordinates of any point on the line be $P(x,y,z) = (x_1 + lr, y_1 + mr, z_1 + nr)$ which lies on the plane ax +by +cz +d = 0,

$a(x_1 + lr) + b(y_1 + mr) + c(z_1 + nr) + d = 0$, □ $r = \frac{ax_1+by_1+cz_1+d}{al+bm+cn}$

substituting this on P(x,y,z) we get point of cross-section.

PERPENDICULAR DISTANCE OF A POINT FORM A LINE:

Let $P(x_2, y_2, z_2)$ be the given point, AB is the line

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r$$

(x_1, y_1, z_1) is fixed point, l,m,n be the D.C's. PM is perpendicular to AB i.e., PM = d.

$$AP=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

AM = projection of AP on AB. (By projection formula)

$$MP^2=AP^2-AM^2=d^2=(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2-$$

$$[\,(x_2-x_1)l+(y_2-y_1)m+(z_2-z_1)n]^2$$

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}\quad .$$